On the number of edges in some graphs ⋆

Chunhui Lai [a,1]

[a] *School of Mathematics and Statistics, Minnan Normal University, Zhangzhou, Fujian, P.R. China.*

Abstract

In 1975, P. Erdős proposed the problem of determining the maximum number $f(n)$ of edges in a graph with n vertices in which any two cycles are of different lengths. The sequence $(c_1, c_2, \cdots, c_n)$ is the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \leq a_i$, where a_i is a nonnegative integer. In 1991, Shi posed the problem of determining $f(a_1, a_2, \cdots, a_n)$ which extended the problem due to Erdős. It is clear that $f(n) = f(1, 1, \cdots, 1)$. Let $g(n, m) = f(a_1, a_2, \cdots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$. We prove that $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2 + \frac{40}{99}}$, which is better than the previous bounds $\sqrt{2}$ (Shi, 1988), and $\sqrt{2 + \frac{7654}{19071}}$ (Lai, 2017). We show that $\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444}$, for all even integers m. We make the following conjecture: $\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}$.

Key words: Graph, cycle, number of edges.
AMS 2000 MSC: 05C38, 05C35.

⋆ Project supported by the National Science Foundation of China (No.61379021; No. 11401290), NSF of Fujian (2015J01018; 2018J01423), Fujian Provincial Training Foundation for "Bai-Quan-Wan Talents Engineering", Project of Fujian Education Department (JZ160455), the Institute of Meteorological Big Data-Digital Fujian and Fujian Key Laboratory of Data Science and Statistics, Project supported by Minnan Normal University.

Email address: `laich2011@msn.cn; laichunhui@mnnu.edu.cn` (Chunhui Lai).
[1] Corresponding author

1 Introduction

Let $f(n)$ be the maximum number of edges in a graph with n vertices in which no two cycles have the same length. In 1975, Erdős raised the problem of determining $f(n)$ (see Bondy and Murty [1], p.247, Problem 11). Shi [11] proved a lower bound.

Theorem 1 (Shi [11])

$$f(n) \geq n + [(\sqrt{8n-23}+1)/2]$$

for $n \geq 3$.

Chen, Lehel, Jacobson and Shreve [3], Jia [4], Lai [5–7], Shi [13,14] obtained some additional related results.

Boros, Caro, Füredi and Yuster [2] proved an upper bound as follows.

Theorem 2 (Boros, Caro, Füredi and Yuster [2]) For n sufficiently large,

$$f(n) < n + 1.98\sqrt{n}.$$

Lai [8] improved the lower bound by Shi as follows.

Theorem 3 (Lai [8]) Let $t = 1260r + 169 \ (r \geq 1)$, then

$$f(n) \geq n + \frac{107}{3}t + \frac{7}{3}$$

for $n \geq \frac{2119}{4}t^2 + 87978t + \frac{15957}{4}$.

Lai [5] proposed the following conjecture:

Conjecture 4 (Lai [5])

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3}.$$

It would be nice to prove that

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \leq \sqrt{3+\frac{3}{5}}.$$

Survey papers on this problem can be found in Tian [15], Zhang [16], Lai and Liu [9].

The progress of all 50 problems in [1] can be found in Locke [10].

The sequence $(c_1, c_2, \cdots, c_n)$ is the cycle length distribution of a graph G with n vertices, where c_i is the number of cycles of length i in G. Let $f(a_1, a_2, \cdots, a_n)$ denote the maximum possible number of edges in a graph which satisfies $c_i \leq a_i$, where a_i is a nonnegative integer. Shi [12] posed the problem of determining $f(a_1, a_2, \cdots, a_n)$ which extended the problem due to Erdős. It is clear that $f(n) = f(1, 1, \cdots, 1)$. Let $g(n, m) = f(a_1, a_2, \cdots, a_n)$, where $a_i = 1$ if i/m is an integer, and $a_i = 0$ otherwise. It is clear that $f(n) = g(n, 1)$.

In this paper, we obtain the following results.

Theorem 5 Let m be even, $s_1 > s_2, s_1 + 3s_2 > k$, then

$$g(n, m) \geq n + (k + s_1 + 2s_2 + 1)t - 1$$

for $n \geq (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1$.

Theorem 6 Let $t = 1260r + 169 \ \ (r \geq 1)$, then

$$f(n) \geq n + \frac{119}{3}t - \frac{26399}{3}$$

for $n \geq \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$.

2 Proof of Theorem 5

Proof. Let $n_t = (\frac{3}{4}mk^2 + \frac{1}{2}mks_1 + \frac{3}{2}mks_2 + \frac{1}{2}ms_1^2 + \frac{3}{2}ms_1s_2 + \frac{9}{4}ms_2^2 + mk + ms_1 + 3ms_2 + \frac{1}{2}m)t^2 + (\frac{1}{4}mk + \frac{1}{2}ms_1 + \frac{3}{4}ms_2 - k - s_1 - 2s_2 + \frac{1}{2}m - 1)t + 1$, m be even, $s_1 > s_2, s_1 + 3s_2 > k$, $n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + (k + s_1 + 2s_2 + 1)t - 1$ edges such that all cycles in G have distinct lengths and all the lengths of cycles are the multiple of m.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq s_1t, i = s_1t + j \ (1 \leq j \leq s_2t), \ i = s_1t + s_2t + j \ (1 \leq j \leq t))$.

Now we define these $B_i s$. These subgraphs all only have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq s_2t$, let the subgraph B_{s_1t+i} consists of a cycle

$$xa_i^1a_i^2...a_i^{ms_1t+2ms_2t+mi-1}x$$

and a path:

$$xa_{i,1}^1a_{i,1}^2...a_{i,1}^{\frac{ms_1t-ms_2t+mi}{2}-1}a_i^{\frac{ms_1t+ms_2t+mi}{2}}.$$

Based on the construction, B_{s_1t+i} contains exactly three cycles of lengths:

$$ms_1t+mi, ms_1t+ms_2t+mi, ms_1t+2ms_2t+mi.$$

For $1 \leq i \leq t$, let the subgraph $B_{s_1t+s_2t+i}$ consists of a cycle

$$C_{s_1t+s_2t+i} = xy_i^1y_i^2...y_i^{ms_1t+3ms_2t+mk(k+1)t+mi-1}x$$

and k paths sharing a common vertex x, the other end vertices are on the cycle $C_{s_1t+s_2t+i}$:

$$xy_{i,p}^1y_{i,p}^2...y_{i,p}^{\frac{ms_1t+3ms_2t-mkt+m(p-1)t+mi}{2}-1}y_i^{\frac{ms_1t+3ms_2t+mk(2p-1)t+m(p-1)t+mi}{2}}(p=1,2,...,k).$$

As a cycle with k chords contains $\binom{k+2}{2}$ distinct cycles, $B_{s_1t+s_2t+i}$ contains exactly $\frac{(k+2)(k+1)}{2}$ cycles of lengths:

$$ms_1t+3ms_2t+mkht+(h+j-1)mt+mi(j \geq 1, h \geq 0, k+1 \geq j+h).$$

B_0 is a path with an end vertex x and length $n-n_t$. The other B_i is simply a cycle of length mi.

Then $g(n,m) \geq n+(k+s_1+2s_2+1)t-1$, for $n \geq n_t$.

This completes the proof.

From Theorem 5, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq$$

$$\sqrt{\frac{(k+s_1+2s_2+1)^2}{(\frac{3}{4}k^2+\frac{1}{2}ks_1+\frac{3}{2}ks_2+\frac{1}{2}s_1^2+\frac{3}{2}s_1s_2+\frac{9}{4}s_2^2+k+s_1+3s_2+\frac{1}{2})}},$$

for all even integers m.

Let $s_1 = 28499066, s_2 = 4749839, k = 14249542$, then

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} > \sqrt{2.444},$$

for all even integers m.

3 Proof of Theorem 6

Proof. Let $n_t = \frac{1309}{2}t^2 - \frac{1349159}{6}t + \frac{6932215}{3}$, $t = 1260r + 169, r \geq 1$, $n \geq n_t$. It suffice to show that there exists a graph G on n vertices with $n + \frac{119}{3}t - \frac{26399}{3}$ edges such that all cycles in G have distinct lengths.

Now we construct the graph G which consists of a number of subgraphs: B_i, $(0 \leq i \leq 22t, i = 22t + j\ (1 \leq j \leq \frac{5t-8}{3})$, $i = 23t + \frac{2t-2}{3} + j\ (1 \leq j \leq \frac{5t-8}{3})$, $i = 32t + j - 60\ (58 \leq j \leq t - 742))$.

Now we define these $B_i s$. These subgraphs all only have a common vertex x, otherwise their vertex sets are pairwise disjoint.

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph B_{22t+i} consists of a cycle

$$xa_i^1a_i^2...a_i^{28t+\frac{2t-2}{3}+2i-3}x$$

and a path:

$$xa_{i,1}^1a_{i,1}^2...a_{i,1}^{\frac{56t-2}{6}}a_i^{\frac{76t-4}{6}+i}.$$

Based on the construction, B_{22t+i} contains exactly three cycles of lengths:

$$22t + i, 25t + \frac{t-1}{3} + i - 1, 28t + \frac{2t-2}{3} + 2i - 2.$$

For $1 \leq i \leq \frac{5t-8}{3}$, let the subgraph $B_{23t+\frac{2t-2}{3}+i}$ consists of a cycle

$$xb_i^1b_i^2...b_i^{28t+\frac{2t-2}{3}+2i-2}x$$

and a path:

$$xb_{i,1}^1b_{i,1}^2...b_{i,1}^{11t-1}b_i^{\frac{76t-4}{6}+i}.$$

Based on the construction, $B_{23t+\frac{2t-2}{3}+i}$ contains exactly three cycles of lengths:

$$23t+\frac{2t-2}{3}+i, 27t+i-1, 28t+\frac{2t-2}{3}+2i-1.$$

For $58 \leq i \leq t-742$, let the subgraph $B_{32t+i-60}$ consists of a cycle

$$C_{32t+i-60}=xy_i^1y_i^2...y_i^{137t+11i+890}x$$

and ten paths sharing a common vertex x, the other end vertices are on the cycle $C_{32t+i-60}$:

$$xy_{i,1}^1y_{i,1}^2...y_{i,1}^{11t-2}y_i^{21t-59+i}$$

$$xy_{i,2}^1y_{i,2}^2...y_{i,2}^{12t-2}y_i^{31t-53+2i}$$

$$xy_{i,3}^1y_{i,3}^2...y_{i,3}^{12t-2}y_i^{41t+156+3i}$$

$$xy_{i,4}^1y_{i,4}^2...y_{i,4}^{13t-2}y_i^{51t+155+4i}$$

$$xy_{i,5}^1y_{i,5}^2...y_{i,5}^{13t-2}y_i^{61t+155+5i}$$

$$xy_{i,6}^1y_{i,6}^2...y_{i,6}^{14t-2}y_i^{71t+154+6i}$$

$$xy_{i,7}^1y_{i,7}^2...y_{i,7}^{14t-2}y_i^{81t+153+7i}$$

$$xy_{i,8}^1y_{i,8}^2...y_{i,8}^{15t-2}y_i^{91t+147+8i}$$

$$xy_{i,9}^1y_{i,9}^2...y_{i,9}^{15t-2}y_i^{101t+149+9i}$$

$$xy_{i,10}^1y_{i,10}^2...y_{i,10}^{16t-2}y_i^{111t+151+10i}.$$

As a cycle with d chords contains $\binom{d+2}{2}$ distinct cycles, $B_{32t+i-60}$ contains exactly 66 cycles of lengths:

$32t+i-60$, $33t+i+4$, $34t+i+207$, $35t+i-3$,
$36t+i-2$, $37t+i-3$, $38t+i-3$, $39t+i-8$,
$40t+i$, $41t+i$, $42t+i+739$, $43t+2i-54$,
$43t+2i+213$, $45t+2i+206$, $45t+2i-3$, $47t+2i-3$,
$47t+2i-4$, $49t+2i-9$, $49t+2i-6$, $51t+2i+2$,
$51t+2i+741$, $53t+3i+155$, $54t+3i+212$, $55t+3i+206$,
$56t+3i-4$, $57t+3i-4$, $58t+3i-10$, $59t+3i-7$,
$60t+3i-4$, $61t+3i+743$, $64t+4i+154$, $64t+4i+212$,
$66t+4i+205$, $66t+4i-5$, $68t+4i-10$, $68t+4i-8$,
$70t+4i-5$, $70t+4i+737$, $74t+5i+154$, $75t+5i+211$,
$76t+5i+204$, $77t+5i-11$, $78t+5i-8$, $79t+5i-6$,
$80t+5i+736$, $85t+6i+153$, $85t+6i+210$, $87t+6i+198$,
$87t+6i-9$, $89t+6i-6$, $89t+6i+735$, $95t+7i+152$,
$96t+7i+204$, $97t+7i+200$, $98t+7i-7$, $99t+7i+735$,
$106t+8i+146$, $106t+8i+206$, $108t+8i+202$, $108t+8i+734$,
$116t+9i+148$, $117t+9i+208$, $118t+9i+943$, $127t+10i+150$,
$127t+10i+949$, $137t+11i+891$.

B_0 is a path with an end vertex x and length $n-n_t$. The other B_i is simply a cycle of length i.

Then $f(n) \geq n+\frac{119}{3}t-\frac{26399}{3}$, for $n \geq n_t$.

This completes the proof.

From Theorem 6, we have

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}},$$

which is better than the previous bounds $\sqrt{2}$ (see [11]), and $\sqrt{2+\frac{7654}{19071}}$ (see [8]).

Combining this with Boros, Caro, Füredi and Yuster's upper bound, namely Theorem 2, we get

$$1.98 \geq \limsup_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

From the proof of Theorem 6, we have

$$\liminf_{n\to\infty} \frac{g(n,m)-n}{\sqrt{\frac{n}{m}}} \geq \sqrt{2+\frac{40}{99}},$$

for all integers m.

If $m=1$, $1\leq i\leq t$, there exists the subgraph similar to $B_{s_1t+s_2t+i}$ consists of a cycle $C_{s_1t+s_2t+i}$ and k paths sharing a common vertex x, the other end vertices are on the cycle $C_{s_1t+s_2t+i}$ such that all cycles in $B_{s_1t+s_2t+i}$ have distinct lengths, then we could obtain

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444} > \sqrt{2+\frac{40}{99}}.$$

But we only for $m=1$, $58\leq i\leq t-742$, construct a subgraph similar to $B_{s_1t+s_2t+i}$ consists of a cycle $C_{s_1t+s_2t+i}$ and ten paths sharing a common vertex x, the other end vertices are on the cycle $C_{s_1t+s_2t+i}$ such that all cycles in $B_{s_1t+s_2t+i}$ have distinct lengths and obtain

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} \geq \sqrt{2+\frac{40}{99}}.$$

Since the liminf for $\frac{g(n,m)-n}{\sqrt{\frac{n}{m}}}$ for even m is $\sqrt{2.444}$, it is reasonable to suspect that such a lower bound also holds for $\frac{f(n)-n}{\sqrt{n}}$.

We make the following conjecture:

Conjecture 7

$$\liminf_{n\to\infty} \frac{f(n)-n}{\sqrt{n}} > \sqrt{2.444}.$$

4 Acknowledgment

The author would like to thank Professor Endre Boros, Yair Caro, Ronald J. Gould, Gyula O.H. Katona, Raphael Yuster for their advice. The author would

like to thank the referees for their many valuable comments and suggestions.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, New York, 1976). MR3138588

[2] E. Boros, Y. Caro, Z. Füredi and R. Yuster, Covering non-uniform hypergraphs, J. Combin. Theory Ser. B 82(2001), 270-284. MR1842115 (2002g:05130)

[3] G. Chen, J. Lehel, M. S. Jacobson and W. E. Shreve, Note on graphs without repeated cycle lengths, J. Graph Theory. 29(1998),11-15. MR1633908 (2000d:05059)

[4] X. Jia, Some extremal problems on cycle distributed graphs, Congr. Numer. 121(1996), 216-222. MR1431994 (97i:05068)

[5] C. Lai, On the size of graphs with all cycle having distinct length, Discrete Math. 122(1993) 363-364. MR1246693(94i:05048)

[6] C. Lai, A lower bound for the number of edges in a graph containing no two cycles of the same length, Electron. J. Combin. 8(2001), Note 9, 1 - 6. MR1877662 (2002k:05124)

[7] C. Lai, Graphs without repeated cycle lengths, Australas. J. Combin. 27(2003), 101-105. MR1955391 (2003m:05102)

[8] C. Lai, On the size of graphs without repeated cycle lengths, Discrete Appl. Math. 232 (2017), 226-229. MR3711962

[9] C. Lai, M. Liu, Some open problems on cycles, J. Combin. Math. Combin. Comput. 91(2014), 51-64. MR3287706

[10] S. C. Locke, Unsolved problems: http://math.fau.edu/locke/Unsolved.htm

[11] Y. Shi, On maximum cycle-distributed graphs, Discrete Math. 71(1988), 57-71. MR0954686 (89i:05169)

[12] Y. Shi, Some problems of cycle length distribution, J. Nanjing Univ. (Natural Sciences), Special Issue On Graph Theory, 27(1991), 233-234.

[13] Y. Shi, The number of edges in a maximum cycle distributed graph, Discrete Math. 104(1992), 205-209. MR1172850 (93d:05083)

[14] Y. Shi, On simple MCD graphs containing a subgraph homemorphic to K_4, Discrete Math. 126(1994), 325-338. MR1264498 (95f:05072)

[15] F. Tian, The progress on some problems in graph theory, Qufu Shifan Daxue Xuebao Ziran Kexue Ban. 1986, no. 2, 30-36. MR0865617

[16] K. Zhang, Progress of some problems in graph theory, J. Math. Res. Exposition 27(3) (2007), 563-576. MR2349503